TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 2 2020

Section 2 Calculator Assumed (**Scientific only**)

3D Vectors

STUDENT'S NAME Presse Solutions

DATE: Friday 8 April **TIME:** 50 minutes **MARKS:** 41

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators (**scientific only**), notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

---

1. (5 marks)

Determine the value(s) of λ if the angle between the vectors $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}1\\0\\\lambda\end{pmatrix}$ is $\frac{\pi}{4}$.

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$\Rightarrow \begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}1\\0\\\lambda\end{pmatrix} = \sqrt{9}\sqrt{1+\lambda^2} \cos\frac{\pi}{4}$$

$$\Rightarrow 1 + 2\lambda = \sqrt{9(1+\lambda^2)} \times \frac{\sqrt{2}}{2}$$

$$\Rightarrow (2+4\lambda)^2 = 18(1+\lambda^2)$$

$$\Rightarrow 4 + 16\lambda + 16\lambda^2 = 18 + 18\lambda^2$$

$$\Rightarrow 0 = 2\lambda^2 - 16\lambda + 14$$

$$\Rightarrow 0 = \lambda^2 - 8\lambda + 7$$

$$\Rightarrow 0 = (\lambda - 1)(\lambda - 7)$$

$$\therefore \lambda = 1 \text{ or } 7$$

2. (12 marks)

Given the two lines L_1 : $\underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and L_2 : $\underset{\sim}{r} = \begin{pmatrix} -1 \\ -5 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$

(a) Show that the lines intersect at a point P and determine the coordinates of P. [4]

Point of intersection

$\lambda = -1 + 2\mu$ (1)

$-1 + 2\lambda = -5 + 6\mu$ (2)

$1 + 2\lambda = 3\mu$ (3)

Sus (1) into (3)

$\Rightarrow \quad 1 + 2(-1+2\mu) = 3\mu$

$\Rightarrow \quad \mu = 1$

$\therefore$ Point P

$$\underset{\sim}{r_P} = \begin{pmatrix} -1 \\ -5 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore$ Coordinates of P is $(1, 1, 3)$

(b) Determine, in the form $ax + by + cz = d$, the equation of the plane containing both the lines L_1 and L_2. [4]

Normal to the plane

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-12 \\ 4-3 \\ 6-4 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$$

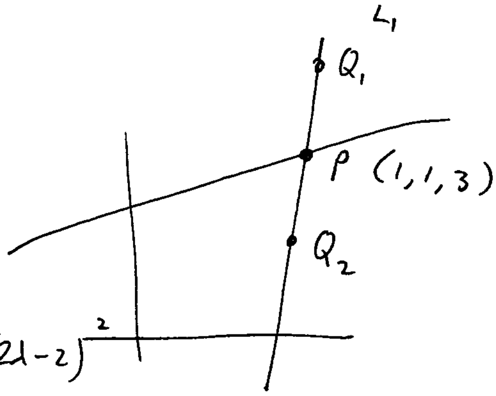
i	j	k	i	j
1	2	2	1	2
2	6	3	2	6

$\therefore$ Plane

$$\underset{\sim}{r} \cdot \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix} = 1$$

$\therefore \quad -6x + y + 2z = 1$

(c) Given that Q_1 and Q_2 are two points on L_1 such that $|PQ_1| = |PQ_2| = 6$, determine the coordinates of Q_1 and Q_2. [4]



Since Q_1 is on L_1, it has coordinates

$(\lambda, -1+2\lambda, 1+2\lambda)$

Now $|PQ_1|^2 = (\lambda-1)^2 + (2\lambda-2)^2 + (2\lambda-2)^2$

$\Rightarrow \quad 36 = \lambda^2 - 2\lambda + 1 + 2(4\lambda^2 - 8\lambda + 4)$

$\vdots$

$\Rightarrow \quad 0 = \lambda^2 - 2\lambda - 3$

$\Rightarrow \quad 0 = (\lambda+1)(\lambda-3)$

$\therefore \lambda = -1$ or 3

$\therefore Q_1 = (-1, -3, -1)$ and $Q_2 = (3, 5, 7)$

OR

Line 1 has eqn

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

the magnitude of the direction is $\sqrt{9} = 3$

$\therefore Q_1$ is 2 vectors of magnitude away

$$\therefore Q_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} = (3, 5, 7)$$

$$Q_2 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = (-1, -3, -1)$$

3. (4 marks)

Two particles A and B have position vectors:

$$\underset{\sim}{r}(t) = (20+10t)\underset{\sim}{i} + 10t\underset{\sim}{j} + (-1.4+0.3t)\underset{\sim}{k} \quad \text{and}$$

$$\underset{\sim}{r}(t) = (30+8t)\underset{\sim}{i} + (-90+10t)\underset{\sim}{j} + (0.9+0.2t)\underset{\sim}{k}$$

Determine whether the particles collide or their paths cross. State the coordinates of this position.

Equate components and use different t values

$$20 + 10t_1 = 30 + 8t_2 \qquad (1)$$

$$10t_1 = -90 + 10t_2 \qquad (2)$$

$$-\frac{14}{10} + \frac{3}{10}t_1 = \frac{9}{10} + \frac{2}{10}t_2 \qquad (3)$$

$(1) - (2)$

$\Rightarrow \quad 20 = 120 - 2t_2$

$\Rightarrow \quad t_2 = 50$

$\therefore \quad t_1 = 41$

$\therefore$ Particles do not collide, but their paths cross.

Point of intersection

$$\begin{pmatrix} 30 + 8(50) \\ -90 + 10(50) \\ 0.9 + 0.2(50) \end{pmatrix}$$

$$= \begin{pmatrix} 430 \\ 410 \\ 10.9 \end{pmatrix}$$

4. (9 marks)

A plane Π contains two lines:

$$\underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}\right) \text{ and } \underset{\sim}{r} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \mu\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$$

(a) Write down a vector equation of the plane Π in the form $\underset{\sim}{r} = \underset{\sim}{a} + \alpha\underset{\sim}{b} + \beta\underset{\sim}{c}$ [1]

$$\underset{\sim}{r} = \begin{pmatrix}1\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}-1\\1\\3\end{pmatrix}$$

(b) The point A with position vector $\begin{pmatrix}8\\2\\c\end{pmatrix}$ lies on the plane. Determine the value of the constant c. [3]

$$\begin{pmatrix}8\\2\\c\end{pmatrix} = \begin{pmatrix}1+2\lambda-\mu\\-1+3\lambda+\mu\\2-\lambda+3\mu\end{pmatrix} \quad \begin{matrix}1\\2\\3\end{matrix}$$

$1+2 \Rightarrow 10 = 5\lambda$

$\lambda = 2$

$\mu = 3-3(2) = -3$

$\therefore c = 2-(2)+3(-3)$

$= -9$

(c) The vector $a\underset{\sim}{i} + b\underset{\sim}{j} + \underset{\sim}{k}$ is perpendicular to the plane. Determine the values of a and b. [3]

$$\begin{pmatrix}2\\3\\-1\end{pmatrix} \times \begin{pmatrix}-1\\1\\3\end{pmatrix} = \begin{pmatrix}9--1\\1-6\\2--3\end{pmatrix}$$

$$\begin{matrix} i & j & k & i & j \\ 2 & 3 & -1 & 2 & 3 \\ -1 & 1 & 3 & -1 & 1 \end{matrix}$$

$$= \begin{pmatrix}10\\-5\\5\end{pmatrix}$$

$$= 5\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$\therefore a = 2$

$b = -1$

(d) State the equation of Π in the form $\underset{\sim}{r} \cdot \underset{\sim}{n} = k$ [2]

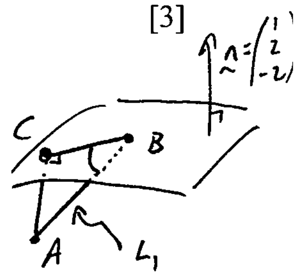
$$\underset{\sim}{r} \cdot \begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix}2\\-1\\1\end{pmatrix} = 6$$

5. (11 marks)

The point A has coordinates $(2,0,-1)$ and the plane Π has the equation $\underset{\sim}{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=8$. The line through A parallel to the line $\underset{\sim}{r}=\begin{pmatrix}0\\0\\-1\end{pmatrix}+\beta\begin{pmatrix}-2\\1\\2\end{pmatrix}$ meets Π at B and the perpendicular from A to Π meets Π in the point C. [Hint: draw a diagram]

(a) Determine the vector equation of the line through A parallel to the given line, hence determine the coordinates of B. [3]



L_1 : $\underset{\sim}{r}_1=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\2\end{pmatrix}$

L_1 intersects plane when

$\begin{pmatrix}2-2\lambda\\\lambda\\-1+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=8$

$\Rightarrow\ 2-2\lambda+2\lambda+2-4\lambda=8$

$\Rightarrow\ \lambda=-1$

$\therefore\ B=\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}-2\\1\\2\end{pmatrix}$

$=\begin{pmatrix}4\\-1\\-3\end{pmatrix}$ or $(4,-1,-3)$

(b) Determine the vector equation of the perpendicular from A to Π, hence show that the coordinates of C are $\left(\frac{22}{9},\frac{8}{9},\frac{-17}{9}\right)$. [3]

L_2 $\underset{\sim}{r}_2=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}1\\2\\-2\end{pmatrix}$

L_2 intersects plane when

$\begin{pmatrix}2+\mu\\2\mu\\-1-2\mu\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=8$

$\Rightarrow\ 2+\mu+4\mu+2+4\mu=8$

$\Rightarrow\ \mu=\frac{4}{9}$

$\therefore\ C=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\frac{4}{9}\begin{pmatrix}1\\2\\-2\end{pmatrix}$

$=\begin{pmatrix}22/9\\8/9\\-17/9\end{pmatrix}$

$=\left(\frac{22}{9},\frac{8}{9},\frac{-17}{9}\right)$

(c) Show that the length of AC is $\frac{4}{3}$. [3]

$$\vec{AC} = \begin{pmatrix} 4/9 \\ 8/9 \\ -8/9 \end{pmatrix}$$

$$= \frac{4}{9}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$$|\vec{AC}| = \frac{4}{9} \times \sqrt{9}$$

$$= \frac{4}{3}$$

(d) Determine the value of $\sin(\angle ABC)$. [2]

$$\sin(\angle ABC) = \frac{|\vec{AC}|}{|\vec{AB}|}$$

$$= \frac{4/3}{3}$$

$$= \frac{4}{9}$$

$$\vec{AB} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

$$|\vec{AB}| = \sqrt{9}$$

$$= 3$$